Intuitionistic and uniform provability in reverse mathematics

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論　文　内　容　要　旨

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The main theme of this thesis is the relationship between intuitionistic and uniform provability in reverse mathematics [S09]. A subsystem RCA of second-order arithmetic, which is obtained by adding full second-order induction scheme to the most popular base system RCA_0 of reverse mathematics, corresponds to non-uniform computable mathematics. In particular, if a Π^1_2 theorem $\forall X(\varphi(X) \rightarrow \exists Y\psi(X,Y))$ is provable in RCA, for all X satisfying $\varphi(X)$, there is an algorithm Φ which computes Y satisfying $\psi(X,Y)$ with the use of X as oracle. However, they may not be a uniform algorithm Φ which computes the witness Y for any oracle X such that $\varphi(X)$. Corresponding to this difference, even if a Π^1_2 theorem $\forall X(\varphi(X) \rightarrow \exists Y\psi(X,Y))$ is provable in RCA, its uniform version $\exists\Phi\forall X(\varphi(X) \rightarrow \psi(X, \Phi(X)))$ or its sequential version $\forall\langle X_n\rangle_{n\in\mathrm{N}}(\forall n\, \varphi(X_n) \rightarrow \exists\langle Y_n\rangle_{n\in\mathrm{N}}\forall n\, \psi(X_n, Y_n))$ may not be provable in RCA^{ω}. On the other hand, the notion of uniform computability is closely related to constructive mathematics [TD88], which is formalized as a system of many-sorted arithmetic based on intuitionistic logic. Historically constructive mathematics has been developed informally in contrast to the formalist foundation of mathematics. Along with the development of reverse mathematics and the discovery of the arithmetical hierarchy of the law-of-excluded-middle principles [ABHK04], however, so-called constructive reverse mathematics [I05], which investigates the interrelations between mathematical statements and logical principles over intuitionistic arithmetic, has been carried out in this decade.

In fact, there are several corresponding results between constructive reverse mathematics and classical reverse mathematics of sequential versions. For example, the principle of trichotomy for reals is intuitionistically equivalent to Σ^0_1-LEM whereas its sequential version

is equivalent to ACA. On the other hand, the principle of dichotomy for reals is intuitionistically equivalent to Σ^0_1-DML whereas its sequential version is equivalent to WKL. More directly, ACA and WKL are intuitionistically equivalent to Σ^0_1-LEM and Σ^0_1-DML respectively in the presence of a choice scheme. Based on these facts, we provide a comprehensive analysis of the connection between intuitionistic provability and classical uniform provability in reverse mathematics. In Chapter 3, we provide a definitive connection between the aforementioned two notions. In particular, we first give an exact formulation to represent uniform provability in RCA and show that for any Π^1_2 formula of some syntactical form (rich enough), it is intuitionistically provable if and only if it is uniformly provable in RCA. The primary tool for the direction from left to right is formalized realizability with functions. The converse direction is shown by using a form of negative translation. In Chapter 4, along the line of the previous [HM11, D14], we study the metatheorems which enable us to apply reverse mathematics to show intuitionistic unprovability. Whereas all of the previous metatheorems are now concerned with sequential versions, our metatheorems are concerned with uniform versions. Applying our metatheorems to the investigation of uniform versions in higher-order reverse mathematics, one can obtain stronger intuitionistic unprovability results than the former case. We use several proof interpretations for the proofs. In Chapter 5, we observe that one has to pay careful attention to the formalization when one considers sequential or uniform versions. Using these results, we show that Dorais' results from [D14] are optimal. In addition, we develop the reverse mathematics of concrete theorems like variants of marriage theorem and symmetric marriage theorem from the perspective of uniformity. Finally, we investigate (over the weak extensional variant of RCA_0^{ω}) the uniform versions of the existence of Jordan decomposition, the principle of trichotomy for reals and Π^0_1 least number principle, which demonstrates that our metatheorems in Chapter 4 are widely applicable to Π^1_2 statements whose sequential versions imply ACA. In Chapter 6, we investigate logical principles weaker than Markov's principle in the context of constructive reverse mathematics. In particular, we provide the complete classification of Π^0_1-DML, Δ^0_1-LEM, Δ^0_1-CA and WMP. However, the corresponding uniform provability in classical
reverse mathematics is still missing.

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論文審査の結果の要旨

本博士論文の目的は直観主義論理における証明可能性と逆数学的観点における一様な証明可能性との関係を明らかにすることである．

この論文ではとくに，存在定理を $\forall X(\varphi(X) \to \exists Y\psi(X, Y))$ という形をしているものとみなし研究を行っている．ここで，φ は扱っている数学的対象 X を規定するものであり，ψ は証明すべき存在 Y の条件を述べたものである．

一般に存在定理の証明を見てみると，あるものの存在を示すときに，単に存在がわかるだけの場合と，具体的な構成方法が与えられている場合がある．これらの本質的違いを証明論的に分析する手段がこれまでにもいくつか考えられてきた．そのうちの２つが逆数学的手段と構成的数学である．構成的数学は，直観主義論理をベースにして数学の定理と排中律などの論理原理の強さとの関係を比べるものであり，逆数学的手段では，存在定理を列化したり，高階算術の下で解を与える汎関数の存在という主張に変えると行った一様化を行って，それらに対応する集合存在原理を調べる．

この２つの手段による結果の類似性はよく知られていたが，明確な数学的事実としての対応関係について研究されることはあまり調べられてこなかった．本博士論文で述べられている研究は，これらの関係をメタ数学的に記述するための定式化を与え，メタ定理として一般的な性質を与えている．

以下では各章にしたがって，そこで得られている結果についてその意義とともに簡単に述べておく．

第 3 章で，考察されるべき二階算術における定理の一様化の正確な形を与え，その RCA における証明可能性が直観主義における証明可能性と対応することを示している．そのために藤原君は，直観主義算術における旧来の方法を精密に分析している．

第 4 章では,逆数学的結果を直観主義体系における証明不可能性に適用するためのメタ定理を扱っている．ここでは高階算術上で与えられる定理の一様版を使って，より強い直観主義体系での証明不可能性を導き出す方法を与えている．

第 5 章では，定理を一様化するときの形式的問題について調べている．とくにここで述べられたことから Dorais の結果が最適であることが導ける．一方で，具体的な逆数学的結果を一様化の視点から見直している．その中には本人が以前に示した結婚定理に関するものを含め，Jordan 分解，実数の順序の線形性などがある．

第 6 章では Markov 原理より弱い論理原理に対して構成的逆数学の観点からの研究を行った．これらは主に排中律型の論理原理に注目したもので，特に，Π^0_1-DML，Δ^0_1-LEM，Δ^0_1-CA，WMP の分類を完成させたことが重要である．

以上のことは彼が自立して研究活動を行うに必要な高度の研能力と学識を有することを示している．したがって，藤原誠君提出の博士論文は，博士（理学）の学位論文として合格と認める．